TMSP 21/22
Problems 1, 14.03.2022
(Please solve each problem on a separate sheet of paper)

1. Consider the one-dimensional Ising model with periodic boundary conditions and $h = 0$. Evaluate the correlation function in the thermodynamic limit:

$$h(k, \ell; T) := \lim_{N\to\infty} \Gamma(k, l; T, N) \tag{1}$$

where

$$\Gamma(k, l; T, N) = \langle s_k\, s_\ell \rangle \; - \; \langle s_k \rangle \; \langle s_\ell \rangle \tag{2}$$

and $\ell > k$. To simplify calculations one may use the spectral decomposition of the transfer matrix $\mathbf{L}$

$$\mathbf{L} = \sum_{i=1}^{2} \lambda_i |v_i >< v_i| = \sum_{i=1}^{2} \lambda_i \, P_i \tag{3}$$

and the relation

$$L_{s_i s_j} =< e_i|\mathbf{L}|e_j > \tag{4}$$

where $|e_i >$ denotes the eigenvector of Pauli matrix σ^z corresponding to eigenvalue s_i.
Is it possible the write the correlation function $h(k, \ell; T)$ in the form

$$h(k, \ell; T) = e^{-\frac{\ell - k}{\xi}}, \tag{5}$$

where $\xi = \left(\log(\tanh K)^{-1}\right)^{-1}$. What is the asymptotic form of ξ for $T \to 0$, i.e., for $K \gg 1$?

2. Consider the one-dimensional Ising model with free boundary conditions and $h = 0$. Rewrite the expression for the partition function $Z_{free}(T, N)$ using the transfer matrix and using the spectral decomposition derive the result obtained already via the recursion formula: $Z_{free}(T, N) = 2\lambda_1^{N-1}$.
Using the above approach find the free energy corresponding to the fixed boundary conditions $s_1 = s_N = 1$ and $s_1 = s_N = -1$. What is the form of the bulk Gibbs free energy density $g_b(T)$ in each of the above cases, i.e., for free, fixed, and periodic b.c.?

3. Consider the one-dimensional XY model with periodic boundary conditions:

$$\mathcal{H}_{XY}(\{\theta_i\}) = -J\sum_{i=1}^{N} \vec{s}_i \cdot \vec{s}_j \tag{6}$$

where the two-dimensional vectors (spins) $\vec{s}_i$ have unit length and can be rewritten as $\vec{s}_i = (s_i^x, s_i^y) = (\cos\theta_i, \sin\theta_i)$. Thus

$$\mathcal{H}_{XY}(\{\theta_i\}) = -J\sum_{i=1}^{N} \cos(\theta_{i+1} - \theta_i), \qquad \theta_{N+1} \equiv \theta_1 \tag{7}$$

Evaluate the partition function $Z(T, N)$

$$Z(T,N) = \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_N e^{K\sum\limits_{i=1}^{N}\cos(\theta_{i+1}-\theta_i)} \tag{8}$$

using the transfer matrix method

$$\mathbf{L}\,|\varphi_p >= \lambda_p\,|\varphi_p >, \tag{9}$$

where $< \theta|\varphi_p >= \varphi_p(\theta), \quad < \theta|\mathbf{L}|\theta' >= e^{K\cos(\theta-\theta')}$ and thus

$$\int_0^{2\pi} d\theta'\, e^{K\cos(\theta-\theta')}\,\varphi_p(\theta') = \lambda_p \varphi_p(\theta) \tag{10}$$

Calculate the free energy per spin in thermodynamic limit, $g_b(T)$, and make the corresponding plot.
Hint: Consider the identity

$$e^{K\cos\psi} \;=\; \sum_{p=-\infty}^{\infty} I_p(K) e^{ip\psi} \tag{11}$$

where $I_n(z)$ is the n-th modified Bessel function [this identity follows from the form of the generating function for modified Bessel functions $\exp\frac{z}{2}(t+\frac{1}{t}) = \sum\limits_{n=-\infty}^{\infty} I_n(z)t^n$] .